LINEARIZED GRAVITY AND GAUGE CONDITIONS

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ABSTRACT. In this paper we consider the field equations for linearized gravity and other integer spin fields on the Kerr spacetime, and more generally on spacetimes of Petrov type D. We give a derivation, using the GHP formalism, of decoupled field equations for the linearized Weyl scalars for all spin weights and identify the gauge source functions occuring in these. For the spin weight 0 Weyl scalar, imposing a generalized harmonic coordinate gauge yields a generalization of the Regge-Wheeler equation. Specializing to the Schwarzschild case, we derive the gauge invariant Regge-Wheeler and Zerilli equation directly from the equation for the spin 0 scalar.

CONTENTS

1. INTRODUCTION

In his 1965 paper [37] Penrose showed that, at least formally, all solutions of the massless spin s field equation on Minkowski space can be obtained from solutions of the spin $(s - \frac{1}{2})$ equations. By repeating this process $2s$ times, one finds that solutions of to the spin 0 equation (i.e. the free scalar wave equation) are potentials for the massless spin s field. This is an example of the spin-raising and -lowering transformations discussed in more detail in [41, section 6.4], see also [41, section 6.7].

The following special case of the construction is relevant here. Given a 2 index Killing spinor, i.e. a spinor K_{AB} solving the equation

$$\nabla_{(A}{}^{A'} K_{BC)} = 0,$$

and a solution $\phi_{ABC...D} = \phi_{(ABC...D)}$ of the spin s zero rest-mass equation

$$\nabla^{AA'} \phi_{ABC...D} = 0,$$

the spin-lowered field $\hat{\phi}_{C...D} = \phi_{ABC...D} K^{AB}$ is a solution of the spin $(s-1)$ zero rest-mass equation.

The analysis of linear field equations on Minkowski space is a key step in the proof of the non-linear stability of Minkowski space [11, 12]. For the case of Minkowski space, the linearized Bianchi equation is precisely the massless spin 2 equation. Thus, the discussion of Penrose shows that the linearized stability problem for the

Einstein equation on Minkowski space can essentially be reduced to a study of the scalar wave equation.

The problem of non-linear stability of Minkowski space, solved in [12], see also [32], can be viewed as a warm-up for the *black hole stability problem*, i.e. the problem of proving the non-linear stability of the Kerr spacetime in the class of asymptotically flat vacuum spacetimes, see e.g. [1, 15, 22], and references therein. The black hole stability problem adds several levels of difficulty over the problem of stability of Minkowski space. The aspect which we shall focus on in the present discussion is that the background spacetime (or more properly stated, the asymptotic state for the evolution) fails to be conformally flat and consequently the relation between the spin 0 wave equation, viewed as a model problem for the full non-linear stability problem, and the equations of linearized gravity fails to be as close as on the Minkowski background.

In particular, the equation for linearized gravity (i.e. the linearized Bianchi system) on a non-conformally flat vacuum background is not the spin-2 system, but has a non-trivial right hand side. In fact, the massless spin 2 equation in a spacetime of Petrov type D has only trivial solutions, cf. [8]. It follows that the spin-lowering and -raising transformations cannot be applied directly to Maxwell or linearized gravity on a background spacetime which is not conformally flat.

However, if we consider the field equations of integer spin on a vacuum type D spacetime, of which Kerr is a special case, then it turns out that an analogue of the spin-lowering transformation does yield useful results, even for the equations of linearized gravity. This idea was alluded to already in the paper of Jeffryes, see [29, p. 340]. Although not stated as explicitly, related ideas play an important role in the work of Fackerell and Crossman [14, 18].

From the point of view of the black hole stability problem, the two-parameter Kerr family of rotating black hole solutions are the vacuum type D spacetimes of most immediate interest. However, most of the results in this paper are valid for general vacuum type D spacetimes. These include the Kerr-Taub-NUT spacetimes, see [45, section 21.1], see also [16].

1.1. **Spin-lowering the Maxwell field.** Recall[1] [50] that any vacuum type D spacetime admits a Killing spinor of the form

$$K_{AB} = \Psi_2^{-1/3} o_{(A} \iota_{B)}. \tag{1.1}$$

A Killing spinor which in addition satisfies the condition

$$\nabla_{A'}{}^B K_{AB} + \nabla_A{}^{B'} \bar{K}_{A'B'} = 0$$

corresponds via

$$K_{ab} = i(K_{AB}\epsilon_{A'B'} - \bar{K}_{A'B'}\epsilon_{AB})$$

to a Killing-Yano tensor, i.e. a skew 2-tensor $K_{ab} = K_{[ab]}$ satisfying the Killing-Yano equation

$$\nabla_{(a} K_{b)c} = 0,$$

see [38, 9, 13, 30] for further information.

Let ϕ_{AB} be the Maxwell spinor, i.e. a solution of the massless spin 1 equation. Then $\phi_{AB}K^{AB} = \Psi_2^{-1/3}\phi_1$, where ϕ_1 is the spin weight 0 Maxwell scalar and this rescaling of ϕ_1 solves a wave equation with potential

$$(\Box + 2\Psi_2)(\Psi_2^{-1/3}\phi_1) = 0, \tag{1.2}$$

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[1] Here and below we use the conventions and notations of the Newman-Penrose (NP) and Geroch-Held-Penrose (GHP) null-tetrad based formalisms, and the associated two-spinor formalism, see [36, 25, 40, 41] and section 2 below.

where $\Box = \nabla^a \nabla_a$ is the d'Alembertian of the spacetime and Ψ_2 is the spin weight 0 Weyl scalar of the background. This is precisely the wave equation derived for the rescaled spin-weight 0 Maxwell scalar on Kerr by Fackerell and Ipser [19], who also argued that ϕ_1 can be used as a potential for the Maxwell field on the Kerr spacetime.

The special case of this spin weight 0 wave equation for the Maxwell field on the Schwarzschild spacetime was used recently in the proof by Blue [6] of decay estimates for the Maxwell equation. In this work, Blue was inspired by Price [42] who showed that the Regge-Wheeler [43] wave equation for axial perturbations of the Schwarzschild spacetime can be viewed as a wave equation, with potential, for a rescaled version of the imaginary part of the spin weight 0 linearized Weyl scalar.

1.2. **Gauge invariant equations for linearized gravity.** We follow the convention of Price and others in discussing perturbations of GHP quantities and let a subindex A denote quantities defined on the background and indicate first order perturbed quantities with a subindex B. See [47, 7] for detailed treatments of perturbation theory in the context of tetrad based formalisms. The Regge-Wheeler wave equation in the form derived by Price can be written as

$$(\Box + 8\Psi_{2A})(\Psi_{2A}^{-2/3} \Im\mathfrak{m}\Psi_{2B}) = 0,$$

on the Schwarzschild spacetime, where in Schwarzschild coordinates $\Psi_{2A} = -Mr^{-3}$.

The approach of Price does not generalize directly to include polar perturbations. These had previously been treated by Zerilli [52], see also Moncrief [35], who derived a non-local wave equation governing a gauge invariant potential for the polar degrees of freedom, and related work by Bičák [3] on perturbations of the Reissner-Nordström solution. Among the difficulties in generalizing Price's approach to the Regge-Wheeler equation to cover general perturbations even of Schwarzschild is that in the background the Ψ_{2A} is non-zero and hence Ψ_{2B} fails to be gauge invariant. In fact, it is only the linearized Weyl scalars Ψ_{0B}, Ψ_{4B} of extreme spin weights $2, -2$, which are both coordinate and tetrad-gauge invariant, and hence it is only those which can be directly viewed as physically measurable quantities.

Teukolsky [48, 49] showed, by calculating in the NP formalism on a Kerr background, working in the principal Kinnersley tetrad, that suitably rescaled versions of the extreme spin weight Maxwell ($s = 1, -1$) and linearized Weyl scalars ($s = 2, -2$) on the Kerr spacetime satisfy decoupled and separable wave equations. The resulting system is usually called the Teukolsky Master Equation (TME).

Shortly after the work of Teukolsky, Ryan [44] showed that the vacuum Teukolsky system for linearized gravity can be derived simply by projecting the Penrose wave equation, i.e. the covariant tensor wave equation

$$\Box R_{abcd} = R_{abef} R_{cd}{}^{ef} + 2(R_{aecf} R_b{}^e{}_d{}^f - R_{aedf} R_b{}^e{}_c{}^f),$$

satisfied by the curvature tensor, on a principal null tetrad, and linearizing. This theme has been taken up and generalized to arbitrary vacuum backgrounds by Bini et al. [4, 5], where all equations for the Ψ's and ϕ's are calculated from components of a generalized de Rham operator acting on the Riemann and Maxwell tensor.

Now consider a general vacuum type D background spacetime. Working in a principal tetrad, let ψ_s be one of the fields $\Psi_{0B}, \Psi_{2A}^{-4/3}\Psi_{4B}, \phi_0, \Psi_{2A}^{-2/3}\phi_2$ (letting s be the spin weight of the field). Further, let

$$B_a = -(\rho n_a - \tau \bar{m}_a).$$

Define a generalized wave operator acting on properly weighted quantities by

$$\boxminus_p = g^{ab}(\Theta_a + pB_a)(\Theta_b + pB_b),$$

where Θ_a is the weighted GHP covariant derivative, see section 2 for details. Then, cf. [4], the TME$_s$ for spin s fields can be written in the form

$$(\boxdot_{2s} - 4s^2\Psi_{2A})\psi_s = 0. \tag{1.3}$$

As shown by Teukolsky, for the Kerr case, equation (1.3) can be separated into radial and angular equations. Due to this fact, it has been possible to formally analyze the Teukolsky system and its solutions and many interesting discoveries have been made. Among these are the relation of the separability of the system to the presence of the Carter constant and corresponding symmetry operators, the calculation of the separation constants, as well as the proof of mode stability for the Teukolsky system [51].

However, it is not clear from these works that the Teukolsky system is well suited for the analysis of the asymptotic decay properties of the higher spin fields. Among the difficulties encountered in attempting to analyze the Teukolsky system are the facts that it has a long-range potential and lower-order terms with slowly decaying coefficients. See [28] and citations therein for discussion.

1.3. **Spin lowering the linearized Weyl field.** Recall that on a vacuum type D spacetime, the spin-lowered Weyl field $\psi_{ABCD}K^{CD}$, where K_{AB} is a Killing spinor, satisfies the Maxwell equation, see [31, §3.8], see also [39]. The same statement holds for the linearized Weyl spinor[2] $(\delta\psi)_{ABCD}$ on Minkowski space, and further in that case lowering the spin by 2 gives $(\delta\psi)_{ABCD}K^{AB}K^{CD}$ which is a solution to the spin 0 wave equation. We now consider the equations satisfied by these fields on a vacuum type D background.

Let $(\delta\psi)_{ABCD}$ be the linearized Weyl spinor on a vacuum type D background, and let K_{AB} be the Killing spinor as in (1.1). Then, expanding the spin 1 field $(\delta\psi)_{ABCD}K^{CD}$ which corresponds to a skew 2-tensor, into weighted scalars gives the rescaled linearized Weyl scalars $\hat{\phi}_{i-1} = \Psi_{2A}^{-1/3}\Psi_{iB}$, $i = 1, 2, 3$, of spin weights $1, 0, -1$ and additional terms arising from linearized tetrad. Thus, by analogy with the above, it is reasonable to suppose that the fields $\hat{\phi}_0 = \Psi_{2A}^{-1/3}\Psi_{1B}$, $\Psi_{2A}^{-2/3}\hat{\phi}_2 = \Psi_{2A}^{-1}\Psi_{3B}$ of spin weights $1, -1$, respectively, satisfy an analogue of the Teukolsky system TME$_s$, for $s = 1, -1$, while the spin weight 0 field $\hat{\phi}_1 = \Psi_{2A}^{-2/3}\Psi_{2B}$ can be expected to satisfy an analogue of the Fackerell-Ipser equation. Further, since the non-extreme linearized Weyl scalars fail to be gauge invariant, one also expects the corresponding equations to contain gauge potential terms.

For the spin weight one case, a calculation, see section 3.2, shows

$$(\boxdot_2 - 4\Psi_2)(\Psi_{2A}^{-1/3}\Psi_{1B}) = -6\Psi_{2A}^{2/3}[(\text{þ}' + 2\rho' - \bar{\rho}')\kappa_B - (\eth' + 2\tau' - \bar{\tau})\sigma_B + 2\Psi_{1B}],$$

The equation satisfied by $\Psi_{2A}^{-1}\Psi_{3B}$ is similar. The right hand side of the equation for Ψ_{1B} corresponds to the right hand sides of the equations for ϕ_{0B} on a charged type D background with Ψ_{2A} playing the role of the spin weight 0 Maxwell scalar ϕ_{1A}, see section 3.2. The analogous statement holds for Ψ_{3B} after applying a prime. As we shall see, the conditions that the right hand sides of the equations for Ψ_{1B}, Ψ_{3B} are zero are tetrad gauge conditions. For the case of the perturbed Maxwell field this corresponds to turning off the background charge, i.e. imposing the condition $\phi_{1A} = 0$. The fact that the just mentioned tetrad gauge conditions can be viewed as the "ghost" of the background charge motivated Chandrasekhar [10, p.240] to use the term *phantom gauge*. In fact, Chandrasekhar showed that by imposing the phantom gauge, the rescalings $\Psi_{2A}^{-1/3}\Psi_{1B}, \Psi_{2A}^{-1}\Psi_{3B}$ satisfy the TME$_s$, for $s = 1, -1$, respectively.

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[2]Here we use a δ to denote linearized quantities, eg. $(\delta\psi)_{ABCD}$ in order to avoid confusion with spinor indices.

We show in section 4 below that the phantom gauge can be viewed as prescribing a gauge source function for the tetrad degrees of freedom along the lines of Friedrich [23], with the linearized Weyl field itself as part of the gauge source. The phantom gauge was studied by Chandrasekhar from a formal point of view only, and the possible implications of this procedure for the hyperbolicity and well-posedness of the linearized Einstein equations were not analyzed in his work. We show here that the phantom gauge condition is compatible with a well posed system of equations for linearized gravity.

Finally, we consider the spin weight 0 linearized Weyl scalar. As the background Ψ_{2A} is non-vanishing, one has that Ψ_{2B} is coordinate gauge dependent (but tetrad-gauge independent). Motivated by the previous discussion we consider the equation satisfied by the spin weight 0 field $\Psi_{2A}^{-2/3}\Psi_{2B}$ obtained by lowering the spin of the the linearized Weyl field by 2. A calculation, cf. section 3.2, shows that this rescaled spin weight 0 linearized Weyl scalar solves the equation

$$(\Box + 8\Psi_2)(\Psi_{2A}^{-2/3}\Psi_{2B}) = -3\Box_B\Psi_{2A}^{1/3}, \tag{1.4}$$

where the right hand side is the first order perturbation of the wave operator, acting on the *background* spin weight zero Weyl scalar. In section 5 we show that, restricting to the Schwarzschild case, equation (1.4) contains all of the information in the Regge-Wheeler and Zerilli-Moncrief systems by giving a direct derivation of these systems starting from (1.4).

The condition that the right hand side of (1.4) vanishes can be viewed as a generalized harmonic coordinate condition. It is worth noting that this gauge condition can be imposed also in the Schwarzschild case. Imposing this generalized harmonic gauge condition, the spin weight zero linearized Weyl scalar satisfies the scalar wave equation

$$(\Box + 8\Psi_{2A})(\Psi_{2A}^{-2/3}\Psi_{2B}) = 0, \tag{1.5}$$

which can be viewed as a generalization of Price's version of the Regge-Wheeler equation not only to the full set of perturbations of Schwarzschild but also to perturbations of Kerr. Lun and Fackerell [33] considered the situation on Schwarzschild and argued formally that by imposing a suitable gauge condition, one obtains equation (1.5) (specialized to the Schwarzschild case).

We further point out that a generalized harmonic gauge condition with a gauge source function involving Ψ_{2B} can also be used to modify the potential in (1.5) so that the equation becomes the Fackerell-Ipser equation. However, as already discussed by Crossman and Fackerell [14, 18] this is possible only in the *rotating case*, and in particular for the Kerr family of spacetimes involves a division by a. As in the case of [33], the discussion in the papers [14, 18] is quite formal and the gauge conditions are there not expressed in terms of gauge source functions. It is interesting to note that the just mentioned work of Crossman and Fackerell took as a starting point the Maxwell equation for the spin-lowered Weyl field in a type D spacetime, and its linearization. This has been carried further in the work of Ferrando et al. [20] where gauge conditions yielding an *exact* Maxwell system for the linearized, spin-lowered Weyl field have been considered.

Now, following the approach taken by Blue for the case of Maxwell on Schwarzschild, where the spin weight zero scalar was used as a potential for the Maxwell field, it is an interesting possibility to use $\Psi_{2A}^{-2/3}\Psi_{2B}$ as a potential for the full linearized gravity system. This would allow one to reduce decay estimates for linearized gravity on Kerr to the study of the scalar wave equation with potential (1.5).

As is indicated by the discussion above, there is a great deal of freedom in using gauge conditions to change the nature of the (tetrad based) linearized Einstein

equations. It is to be expected that this remark applies equally to the full, non-linear system of Einstein equations. The implications of this remain to be considered. One could in principle go further, and make use of the gauge dependence of Ψ_{2B} to remove the potential from the equation and achieve a setup (in the rotating case) where $\Psi_{2A}^{-2/3}\Psi_{2B}$ solves the scalar wave equation $\Box(\Psi_{2A}^{-2/3}\Psi_{2B}) = 0$. We point out that the gauge conditions chosen by Chandrasekhar, cf. [10, §82], in his considerations of linearized gravity on the Kerr background included the conditions $\Psi_{1B} = \Psi_{2B} = \Psi_{3B} = 0$. This type of gauge condition will not be considered in detail here.

For linearized gravity on Kerr, among the questions which should be considered are the choice of potential for linearized gravity and the field equation governing this. For the gauge invariant scalars Ψ_{0B}, Ψ_{4B} these issues have been extensively discussed in the literature. Making use of gauge conditions as discussed above opens up interesting new possibilities. However to make full use of these, the problem of reconstructing the full solution of linearized gravity from e.g. Ψ_{2B} must be considered.

1.4. **Overview of this paper.** The plan of this paper is as follows. In section 2 we set up notation and give a brief overview of the GHP formalism and its specialization to vacuum type D backgrounds. We also discuss there gauge issues that arise when working in a tetrad based formalism. Section 3 starts by introducing the properly weighted generalized wave operators which occur in the GHP formalism and in the Teukolsky system. Further, we give there a derivation in the GHP formalism of the equations for linearized gravity, cf. section 3.2. The gauge nature of the non-trivial right hand sides of the equations for the scalars of non-extreme spin weights is discussed in section 4 where we also give gauge-fixed versions of these systems which lead to new potentials for linearized gravity satisfying well-posed field equations. In section 5 the gauge invariant Regge-Wheeler and Zerilli equations for Schwarzschild background are derived from (1.4).

2. Preliminaries and notation

We use the conventions and notations of [25]. In particular we use abstract index notation with lower case latin indices for tensors and upper case latin indices for spinors. For tetrad indices we use lower case fraktur font, while for coordinate indices we use lower case greek letters. Unless otherwise stated we shall consider only vacuum spacetimes (M, g_{ab}) of dimension 4, with signature $+ - - -$.

2.1. **GHP formalism.** The Geroch-Held-Penrose (GHP) null-tetrad formalism [25] allows one to represent the Einstein equations in a compact form, and gives an efficient tool for calculations. Since we will make heavy use of this formalism and its properties, we give, in order to make the paper reasonably self-contained, a brief description of its main features.

Consider a null tetrad $(e_{\mathfrak{a}}^a) = (l^a, n^a, m^a, \bar{m}^a)$ consisting of two real null vectors l^a, n^a and two complex linear combinations of spatial vectors $m^a, \bar{m}^a$, normalized such that the only non-vanishing inner products of the tetrad vectors are

$$l^a n_a = -m^a \bar{m}_a = 1.$$

In order to avoid clutter we suppress the abstract index on the tetrad vectors $e_{\mathfrak{a}}$ when convenient. The coframe $e^{\mathfrak{a}}$ is defined by the relations $e^{\mathfrak{a}}(e_{\mathfrak{b}}) = \delta^{\mathfrak{a}}{}_{\mathfrak{b}}$. We note the useful relations

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - \bar{m}_a m_b, \tag{2.1a}$$

$$\delta^a{}_b = l^a n_b + n^a l_b - m^a \bar{m}_b - \bar{m}^a m_b. \tag{2.1b}$$

A choice of a null tetrad picks out a 2-dimensional subgroup of the Lorentz group in each tangent space (and hence also a reduction of the principal $SO_+(3,1)$ bundle of (M, g), see below) which preserves the null planes spanned by l^a, n^a and the spatial planes spanned by $m^a, \bar{m}^a$. These can be represented in terms of a non-vanishing complex field λ by the *boost rotations*

$$l^a \to \lambda \bar{\lambda} l^a, \quad n^a \to \lambda^{-1} \bar{\lambda}^{-1} n^a, \tag{2.2a}$$

and the *spin rotations*

$$m^a \to \lambda \bar{\lambda}^{-1} m^a, \quad \bar{m}^a \to \lambda^{-1} \bar{\lambda} \bar{m}^a. \tag{2.2b}$$

Projecting tensor fields on the spacetime on the null tetrad gives a representation of these fields in terms of collections of tetrad components which are simply complex fields on the spacetime. In general, a scalar field η defined by projecting a tensor field will transform as

$$\eta \to \lambda^p \bar{\lambda}^q \eta \,,$$

for some integers p, q, under the above defined action of λ. A quantity η which transforms according to the above rule is said to have *type* $\{p, q\}$ and fields with well defined type are referred to as *weighted quantities*. Note the notion of weighted quantity extends to tensors. In particular, the tetrad vectors $l^a, n^a, m^a, \bar{m}^a$ have types $\{1, 1\}$, $\{-1, -1\}$, $\{1, -1\}$, $\{-1, 1\}$, respectively. It is useful to note that the type is additive under multiplication and hence the weighted quantities form a graded algebra. The spin and boost weights s, r of a weighted quantity η are $s = \frac{1}{2}(p - q)$, $r = \frac{1}{2}(p + q)$.

The following formal operations take weighted quantities to weighted quantities,

$$\begin{aligned}
{}^{-}(\text{bar}) &: l^a \to l^a,\ n^a \to n^a,\ m^a \to \bar{m}^a,\ \bar{m}^a \to m^a, & \{p, q\} \to \{q, p\}, \\
{}'(\text{prime}) &: l^a \to n^a,\ n^a \to l^a,\ m^a \to \bar{m}^a,\ \bar{m}^a \to m^a, & \{p, q\} \to \{-p, -q\}, \\
{}^{*}(\text{star}) &: l^a \to m^a,\ n^a \to -\bar{m}^a,\ m^a \to -l^a,\ \bar{m}^a \to n^a, & \{p, q\} \to \{p, -q\}.
\end{aligned} \tag{2.3}$$

The bar $^{-}$ operation acting on a weighted quantity is simply the complex conjugation of the field. We have $\bar{\bar{\eta}} = \eta$ and $\eta'' = (-1)^{p+q} \eta$ (note however that we shall consider only fields with $p + q$ even). Further, the bar $^{-}$ and prime $'$ operations commute, while the star * operation commutes with neither of these. Thus, the star operation has to be treated separately from the bar and prime operations.

For the case of an orthonormalized tetrad, the Levi-Civita connection can be represented in terms of 24 independent real connection coefficients. In terms of the null tetrad introduced above, the connection coefficients $\Gamma^{\mathfrak{c}}_{\mathfrak{ab}} = e^{\mathfrak{c}}_b (\nabla_{e_a} e^b_{\mathfrak{b}})$ combine into 12 complex scalars, called *spin coefficients*. Of these only 8 are properly weighted, and can be represented in terms of the quantities

$$\kappa = m^b l^a \nabla_a l_b, \quad \sigma = m^b m^a \nabla_a l_b, \quad \rho = m^b \bar{m}^a \nabla_a l_b, \quad \tau = m^b n^a \nabla_a l_b, \tag{2.4}$$

and their primes $\kappa', \sigma', \rho', \tau'$. The effect of the star operation on the spin coefficients and their primes and complex conjugates can be calculated directly from (2.4), or see [25, p. 878] for a list. The types of the spin coefficients are

$$\kappa : \{3, 1\}, \quad \sigma : \{3, -1\}, \quad \rho : \{1, 1\}, \quad \tau : \{1, -1\},$$

and the types of their primes are given according to (2.3).

The remaining 4 spin coefficients

$$\beta = \frac{1}{2}(n^b m^a \nabla_a l_b - \bar{m}^b m^a \nabla_a m_b),$$

$$\epsilon = \frac{1}{2}(n^b l^a \nabla_a l_b - \bar{m}^b l^a \nabla_a m_b),$$

and their primes β', ϵ', are not properly weighted and are not used explicitly in the GHP formalism.

Let $\mathbb{C}_*$ denote the non-zero complex numbers. As discussed by Ehlers [17], see also [26, 27], the choice of two null directions gives a reduction of the principal $SO_+(3,1)$ frame bundle of the spacetime to a principal $\mathbb{C}_*$ bundle B with the action of $\mathbb{C}_*$ $z : (\lambda, e_{\mathfrak{a}}) \mapsto e_{\mathfrak{a}}.z(\lambda)$ given by (2.2). The weighted quantities may be viewed as sections of associated complex line bundles[3] $\mathcal{E}^{\{p,q\}} = B \times_{z^{\{p,q\}}} \mathbb{C}$ determined by the representation $z^{\{p,q\}} : (\lambda, v) \rightarrow \lambda^p \bar{\lambda}^q v$ of $\mathbb{C}_*$ on $\mathbb{C}$. The restriction of the Levi-Civita connection to the reduced bundle B induces a connection on the bundles $\mathcal{E}^{\{p,q\}}$ given by

$$\Theta_a \eta = \nabla_a \eta - p \omega_a \eta - q \bar{\omega}_a \eta, \qquad (2.5)$$

where ω_a is the connection form

$$\omega_a = -\epsilon' l_a + \epsilon n_a + \beta' m_a - \beta \bar{m}_a$$
$$= \frac{1}{2}(n^b \nabla_a l_b + m^b \nabla_a \bar{m}_b).$$

Under a gauge transformation ω_a transforms as

$$\omega_a.z = \omega_a + \frac{\nabla_a \lambda}{\lambda},$$

Note that $\omega'_a = -\omega_a$. It follows that

$$(\Theta_a \eta)' = \Theta_a \eta',$$

for any properly weighted quantity η.

The GHP operators $\þ, \þ', \eth, \eth'$ are defined as the Θ_a covariant derivative along the tetrad vectors,

$$\þ = l^a \Theta_a, \quad \þ' = n^a \Theta_a, \quad \eth = m^a \Theta_a, \quad \eth' = \bar{m}^a \Theta_a.$$

The action of the bar, prime and star operations on the GHP operators follows from their action on the tetrad vectors. Expanding Θ_a in terms of the GHP operators gives

$$\Theta_a = l_a \þ' + n_a \þ - m_a \eth' - \bar{m}_a \eth. \qquad (2.6)$$

In terms of the graded algebra of weighted quantities, the covariant derivative Θ_a as well as the GHP operators satisfy a (graded) Leibniz rule. Further, as remarked above, the notions of properly weighted quantity extend to differential forms and more general objects, and hence also the weighted covariant derivative lifts to act on such objects. In particular, the tetrad elements themselves are properly weighted quantities, and the action of Θ_a and the GHP operators on these can be read off from the definitions. We have for example $\þ l_c = \delta_c{}^b l^a \Theta_a l_b$. Expanding this out using (2.1b), (2.5) and the definitions of the spin-coefficients gives $\þ l_c = -\bar{\kappa} m_c - \kappa \bar{m}_c$. The following equations

$$\þ l_a = -\bar{\kappa} m_a - \kappa \bar{m}_a, \qquad\qquad \þ m_a = -\bar{\tau}' l_a - \kappa n_a, \qquad (2.7a)$$
$$\þ' l_a = -\bar{\tau} m_a - \tau \bar{m}_a, \qquad\qquad \þ' m_a = -\bar{\kappa}' l_a - \tau n_a, \qquad (2.7b)$$
$$\eth l_a = -\bar{\rho} m_a - \sigma \bar{m}_a, \qquad\qquad \eth m_a = -\bar{\sigma}' l_a - \sigma n_a, \qquad (2.7c)$$
$$\eth' l_a = -\bar{\sigma} m_a - \rho \bar{m}_a, \qquad\qquad \eth' m_a = -\bar{\rho}' l_a - \rho n_a. \qquad (2.7d)$$

and their primes and complex conjugates give the complete set of relations.

The Leibnitz rule for covariant derivative together with (2.7) allows one to read off the action of the GHP operators on tensors projected on any combination of tetrad vectors, and hence covariant tensor equations can be expressed equivalently as collections of scalar equations in the GHP formalism.

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[3]By the same construction we can also treat e.g. differential forms as weighted quantities.

The 10 degrees of freedom of the Weyl tensor can be represented by the five weighted Weyl scalars

$$\Psi_0 = W_{abcd}m^a l^b l^c m^d, \quad \Psi_1 = W_{abcd}n^a l^b l^c m^d, \quad \Psi_2 = W_{abcd}m^a l^b \bar{m}^c n^d,$$

$$\Psi_3 = W_{abcd}n^a l^b \bar{m}^c n^d, \quad \Psi_4 = W_{abcd}\bar{m}^a n^b n^c \bar{m}^d.$$

Similarly, the Maxwell field strength can be represented by the 3 Maxwell scalars

$$\phi_0 = F_{ab}l^a m^b, \quad \phi_1 = \frac{1}{2}(F_{ab}l^a n^b + F_{ab}\bar{m}^a m^b), \quad \phi_2 = F_{ab}\bar{m}^a n^b.$$

We shall refer to the spin coefficients and the Weyl and Maxwell scalars collectively as GHP quantities.

The Weyl scalars Ψ_i, $i = 0, \ldots, 4$ have types $\{4 - 2i, 0\}$ while the Maxwell scalars ϕ_i, $i = 0, 1, 2$ have types $\{2 - 2i, 0\}$. The prime operation gives $\Psi_i' = \Psi_{4-i}$, $i = 0, \ldots, 4$, and $\phi_i' = -\phi_{2-i}$, $i = 0, 1, 2$. Further, $\Psi_i^* = \Psi_i$ and $\phi_i^* = \phi_i$.

We now state, modulo prime and star operations, the Einstein, Bianchi and Maxwell equations in GHP notation, specialized to the vacuum case. Working in a tetrad formalism, the Einstein equation takes the form of a system of first-order equations for the connection coefficients, i.e. in the GHP setting, for the spin coefficients. These are given by

$$\eth\,\rho - \eth'\sigma = (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \Psi_1, \tag{2.8a}$$

$$\flat\,\rho - \eth'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau', \tag{2.8b}$$

$$\flat\,\sigma - \eth\,\kappa = (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \tag{2.8c}$$

$$\flat\,\rho' - \eth\,\tau' = \rho'\bar{\rho} + \sigma\sigma' - \tau'\bar{\tau}' - \kappa\kappa' - \Psi_2, \tag{2.8d}$$

together with their primed and starred versions. The Bianchi equations are given by

$$(\flat - 4\rho)\Psi_1 - (\eth' - \tau')\Psi_0 = -3\kappa\Psi_2, \tag{2.9a}$$

$$(\flat - 3\rho)\Psi_2 - (\eth' - 2\tau')\Psi_1 = \sigma'\Psi_0 - 2\kappa\Psi_3, \tag{2.9b}$$

together with their primed and starred versions, and the Maxwell equations are

$$(\flat - 2\rho)\phi_1 - (\eth' - \tau')\phi_0 = -\kappa\phi_2, \tag{2.10}$$

with its primed and starred versions. Further, the GHP operators acting on weighted quantities satisfy the commutator relations

$$\big[\flat, \flat'\big]\eta = \big[(\bar{\tau} - \tau')\,\eth + (\tau - \bar{\tau}')\eth' - p(\kappa\kappa' - \tau\tau' + \Psi_2) - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2)\big]\eta, \tag{2.11a}$$

$$\big[\flat, \eth\big]\eta = \big[-\bar{\tau}'\flat - \kappa\flat' + \bar{\rho}\,\eth + \sigma\eth' - p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}')\big]\eta, \tag{2.11b}$$

together with their primed and starred versions.

2.2. **Petrov type D spacetimes.** In a spacetime of type D we can fix a null tetrad up to rescalings (and a trivial rearrangement) by aligning the real null vectors with the principal null directions. Such a tetrad is called a principal tetrad. In a vacuum type D spacetime, working in a principal tetrad, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ follows and due to the Goldberg-Sachs theorem $\kappa = \kappa' = \sigma = \sigma' = 0$. The only non-vanishing GHP quantities are

$$\Psi_2, \rho, \tau, \rho', \tau',$$

and the Bianchi identities (2.9) simplify to

$$\flat\Psi_2 = 3\rho\Psi_2, \quad \eth\Psi_2 = 3\tau\Psi_2, \tag{2.12}$$

together with their primed versions. See [16] for further identities valid for the GHP quantities valid in vacuum type D spacetimes. We record for later use a

commutation relation for $\{p, 0\}$ quantities η on vacuum type D backgrounds. The following identity (and its prime) is a consequence of (2.11b),(2.8a) and (2.8a)*,

$$\left[\text{þ} -a\rho, \eth -a\tau\right]\eta = \bar{\rho}(\eth -a\tau)\eta - \bar{\tau}'(\text{þ} -a\rho)\eta, \tag{2.13}$$

see [21, equation (2.5)] for a more general relation involving q-weight.

2.3. Perturbation theory and gauge transformations.
Here we give a short overview of gauge transformations in perturbation theory. See [47, 7] for more details. Perturbations of a spacetime can be understood in terms of curves in the space of solutions of Einstein field equations, originating at a given background spacetime. Linear perturbations are tangents to such curves at the origin. We denote the perturbation parameter by ϵ.

The identification of points of background and perturbed spacetime is called identification gauge. Introducing coordinates x^a in the background, an infinitesimal transformation of the form $x^a \to x^a + \epsilon \xi^a$ can be interpreted as changing the identification of points between background and perturbed spacetime. Quantities which do not change under these transformations are called identification or coordinate gauge invariant. A quantity is *coordinate gauge invariant* if and only if it vanishes or is a constant scalar or a constant linear combination of products of Kronecker deltas in the background, see [46, p.24].

The Weyl scalars Ψ_{iB} transform as scalars under coordinate transformations,

$$\Psi_{iB} \to \Psi_{iB} - \epsilon \, \xi^\mu \partial_\mu \Psi_{iA} + O(\epsilon^2).$$

In a type D background, only Ψ_2 is non-zero, and hence the spin 0 scalar is the only one of the Ψ_{iB}, $i = 0, \ldots, 4$ which fails to be coordinate gauge invariant.

As mentioned in section 2.1, we choose the background tetrad to be fixed up to a two-dimensional subgroup of the Lorentz group corresponding to boost rotations of the future pointing null vectors l^a, n^a, and spin rotations of $m^a, \bar{m}^a$. However, the perturbed tetrad has the full transformation freedom under infinitesimal elements of the Lorentz group

$$\begin{aligned}
&l_B^a \to l_B^a, &&n_B^a \to n_B^a + \epsilon(\bar{a}m^a + a\bar{m}^a), &&m_B^a \to m_B^a + \epsilon a l^a, \\
&l_B^a \to l_B^a + \epsilon(\bar{b}m^a + b\bar{m}^a), &&n_B^a \to n_B^a, &&m_B^a \to m_B^a + \epsilon b n^a, \\
&l_B^a \to l_B^a + \epsilon A l^a, &&n_B^a \to n_B^a - \epsilon A n^a, &&m_B^a \to m_B^a - \mathrm{i}\epsilon\Theta m^a,
\end{aligned}$$

where a, b are complex and A, Θ are real functions (see e.g. the linearized versions of (2.2) for the third line. A quantity which is invariant under these transformations will be called *tetrad gauge invariant*. For the first subset of infinitesimal Lorentz transformations we have for example

$$\Psi_{jB} \to \Psi_{jB} + \epsilon j \bar{a} \Psi_{j-1\,A}, \quad j = 0, ...4; \Psi_{-1\,A} = 0. \tag{2.14}$$

A complete table for all GHP quantities can be found in [7, §5.10]. It can be verified that Ψ_{0B}, Ψ_{2B} and Ψ_{4B} are tetrad gauge invariant.

In the following, in order to avoid clutter in the notation, we will drop the index A for background quantity, unless it is not clear from the context whether a certain quantity is evaluated on the background.

3. Equations for linearized gravity and electromagnetism

In this section, we derive equations for linear perturbations of the Weyl components $\Psi_0, ..., \Psi_4$ on vacuum type D backgrounds, as well as for the linearized Maxwell scalars ϕ_0, ϕ_1, ϕ_2 on charged type D backgrounds. The gauge invariant fields Ψ_{0B}, Ψ_{4B} satisfy the Teukolsky system TME$_s$, for $s = 2, -2$, while the equations for the tetrad gauge dependent scalars Ψ_{1B}, Ψ_{3B} correspond to the TME$_s$ for $s = 1, -1$, but with a non-trivial right hand side involving a gauge source function, cf. section 4. For the spin weight zero linearized Weyl scalar Ψ_{2B} we find a new

wave equation with a non-trivial right hand side. If the right hand side vanishes, this equation is a direct generalization of the Regge-Wheeler wave equation to type D. In section 4 below we consider the structure of these right hand sides in more detail.

We point out that while the equations for the linearized Weyl scalars as well as the linearized Maxwell scalars decouple in the sense that each individual equation involves only one of these scalars, the equations for the non-extreme spins are coupled via linearized spin coefficients, unless further gauge conditions are imposed.

3.1. **Weighted wave operators.** As is the case for any vector bundle over (M, g) with covariant derivative, there is a natural generalized wave operator acting on sections of the bundles $\mathcal{E}^{\{p,q\}}$. The Weyl and Maxwell scalars are properly weighted quantities of type $\{2s, 0\}$ for integer spin weights s. Since we shall be interested in operators acting on the Weyl and Maxwell scalars, we restrict our attention to the operator $\Box_p = \Theta^a \Theta_a$ acting on quantities of type $\{p, 0\}$. Expanding this using (2.6) and (2.7) gives after some calculations using the commutation relations (2.11a),(2.11b)

$$\Box_p = 2\Big[(\text{þ} - \bar{\rho})(\text{þ}' - \rho') - (\eth - \bar{\tau}')(\eth' - \tau') + \sigma\sigma' - \kappa'\kappa - \Psi_2 \tag{3.1}$$

$$+ \frac{p}{2}(\kappa\kappa' - \tau\tau' + \rho\rho' - \sigma\sigma' + 2\Psi_2)\Big], \tag{3.2}$$

Let ω_a be the connection form in Θ_a, cf. section 2.1, and let B_a be a properly weighted form of type $\{0, 0\}$. Then $\omega_a - B_a$ is again a connection form on the weigted bundles $\mathcal{E}^{\{p,q\}}$ and $\Theta_a + pB_a + q\bar{B}_a$ is again a weighted covariant derivative on the bundles $\mathcal{E}^{\{p,q\}}$. In particular, let

$$B_a = -(\rho n_a - \tau \bar{m}_a). \tag{3.3}$$

Modifying the covariant derivative with B_a gives the weighted wave operator on type $\{p, 0\}$ quantities

$$\boxplus_p = (\Theta^a + pB^a)(\Theta_a + pB_a).$$

Note that $\boxplus_0 = \Box$. We can now write the vacuum Teukolsky master equation for a spin weight s field $\psi^{(s)}$ in the form

$$[\boxplus_{2s} - 4s^2\Psi_2]\psi^{(s)} = 0,$$

cf. Bini et al. [5, §4].

A calculation shows that acting on a quantity of type $\{p, 0\}$ we have

$$\boxplus_p = \Box_p + 2pB^a\Theta_a + p(\Theta^a B_a) + p^2 B^a B_a$$
$$= 2(\text{þ} - p\rho - \bar{\rho})(\text{þ}' - \rho') - 2(\eth - p\tau - \bar{\tau}')(\eth' - \tau')$$
$$+ (p - 2)[\kappa\kappa' - \sigma\sigma'] + (3p - 2)\Psi_2. \tag{3.4}$$

Restricting to a type D background we have

$$\boxplus_p = 2(\text{þ} - p\rho - \bar{\rho})(\text{þ}' - \rho') - 2(\eth - p\tau - \bar{\tau}')(\eth' - \tau') + (3p - 2)\Psi_2. \tag{3.5}$$

since κ, σ vanish there.

Recalling the discussion in section 2.1, Θ_a transforms properly under the prime operation. In particular, we have $(\Theta_a\eta)' = \Theta_a\eta'$ and hence also $(\Theta^a\Theta_a\eta)' = \Theta^a\Theta_a\eta'$ for any properly weighted quantity. However, modified connection $\Theta_a + pB_a$ does not satisfy this rule since $B_a' \neq -B_a$. Instead, the operator $\boxplus_p$ has the following transformation rule involving rescalings.

Lemma 3.1. *Let η be a properly weigthed quantity of type $\{p,0\}$. The generalized wave operator $\boxdot_p$ on a vacuum type D background transforms under prime as*

$$(\boxdot_p \eta)' = \Psi_2^{-p'/3} \boxdot_{p'} (\Psi_2^{p'/3} \eta'). \tag{3.6}$$

Remark 3.2. *It should be noted that the p in $\boxdot_p$ denotes the weight of the quantity on which it acts. Hence the p' in the right hand side of (3.6) is $p' = -p$, since the type of η' is $\{-p,0\}$.*

Proof. Using the commutation relations and field equations on type D, one gets the identity

$$(\text{\th}' + p\rho' - \bar{\rho}')(\text{\th} - \rho) - (\eth' + p\tau' - \bar{\tau})(\eth - \tau)$$
$$= (\text{\th} - \bar{\rho})(\text{\th}' + (p-1)\rho') - (\eth - \bar{\tau}')(\eth' + (p-1)\tau') + 3p\Psi_2. \tag{3.7}$$

Rescaling the RHS by $\Psi_2^{p/3}$, using Bianchi identities (2.12) gives the result. $\square$

As we shall see, using this transformation property for $\boxdot_p$, half of the equations for linearized gravity discussed below follow without calculation.

3.2. **Perturbation calculations.** We now derive the equations for the linearized Weyl scalars in terms of the weighted wave operators.

Theorem 3.3. *On a vacuum type D background we have*

$$\left[\boxdot_4 - 16\Psi_2\right]\Psi_{0B} = 0, \tag{3.8}$$

$$\left[\boxdot_2 - 4\Psi_2\right]\left(\Psi_2^{-1/3}\Psi_{1B}\right) = -6\Psi_{2A}^{2/3}[(\text{\th}' + 2\rho' - \bar{\rho}')\kappa_B - (\eth' + 2\tau' - \bar{\tau})\sigma_B + 2\Psi_{1B}], \tag{3.9}$$

$$\left[\boxdot_0 + 8\Psi_2\right]\left(\Psi_2^{-2/3}\Psi_{2B}\right) = -3\square_B\Psi_2^{1/3}, \tag{3.10}$$

$$\left[\boxdot_{-2} - 4\Psi_2\right]\left(\Psi_2^{-1}\Psi_{3B}\right) = -6[(\text{\th} + 2\rho - \bar{\rho})\kappa'_B - (\eth + 2\tau - \bar{\tau}')\sigma'_B + 2\Psi_{3B}] \tag{3.11}$$

$$\left[\boxdot_{-4} - 16\Psi_2\right]\left(\Psi_2^{-4/3}\Psi_{4B}\right) = 0, \tag{3.12}$$

Proof. First we consider the equations for linearized the Weyl scalars Ψ_{0B}, Ψ_{4B} with extreme spin weights $s = 2, -2$. The linearized Bianchi identities $(2.9a)^*$ and $(2.9a)$ read

$$(\text{\th}' - \rho')\Psi_{0B} = (\eth - 4\tau)\Psi_{1B} + 3\sigma_B\Psi_2, \tag{3.13}$$

$$(\eth' - \tau')\Psi_{0B} = (\text{\th} - 4\rho)\Psi_{1B} + 3\kappa_B\Psi_2. \tag{3.14}$$

Combining these identities as $(\text{\th} - 4\rho - \bar{\rho})(3.13) - (\eth - 4\tau - \bar{\tau}')(3.14)$ gives

$$\left[(\text{\th} - 4\rho - \bar{\rho})(\text{\th}' - \rho') - (\eth - 4\tau - \bar{\tau}')(\eth' - \tau')\right]\Psi_{0B} =$$
$$\left[(\text{\th} - 4\rho - \bar{\rho})(\eth - 4\tau) - (\eth - 4\tau - \bar{\tau}')(\text{\th} - 4\rho)\right]\Psi_{1B} \tag{3.15}$$
$$+ 3\left[(\text{\th} - 4\rho - \bar{\rho})\sigma_B - (\eth - 4\tau - \bar{\tau}')\kappa_B\right]\Psi_2.$$

The term involving Ψ_{1B} on the RHS vanishes due to (2.13) with $a = 4$. The perturbed Ricci identity (2.8c) reads $(\text{\th} - \rho - \bar{\rho})\sigma_B - (\eth - \tau - \bar{\tau}')\kappa_B = \Psi_{0B}$ and from (2.12) it follows, that also the Ψ_2 term on the RHS reduces to $3\Psi_{0B}\Psi_2$. Recalling the form of $\boxdot_p$ gives equation (3.8) for Ψ_{0B}. Equation (3.12) for Ψ_{4B} follows from this after applying a prime and using (3.6).

As we shall see, the corresponding wave equations governing the linearized Weyl scalars Ψ_{1B}, Ψ_{3B} with spin weights $s = 1, -1$ do not decouple in the sense that

other perturbed quantities than the given linearized Weyl scalar are involved. The linearized Bianchi identities (2.9b)* and (2.9b)$'$ read

$$(\text{þ}' - 2\rho')\Psi_{1B} = \left\{(\text{ð}-3\tau)\Psi_2\right\}_B,$$

$$-(\text{ð}' - 2\tau')\Psi_{1B} = -\left\{(\text{þ}-3\rho)\Psi_2\right\}_B,$$

(3.16)

Multiplying both equations by $\Psi_2^{-1/3}$ and using the Leibniz rule gives

$$(\text{þ}' - \rho')(\Psi_2^{-1/3}\Psi_{1B}) = \frac{3}{2}\left\{(\text{ð}-2\tau)\Psi_2^{2/3}\right\}_B, \qquad (3.17)$$

$$-(\text{ð}' - \tau')(\Psi_2^{-1/3}\Psi_{1B}) = -\frac{3}{2}\left\{(\text{þ}-2\rho)\Psi_2^{2/3}\right\}_B. \qquad (3.18)$$

Here we have made use of the fact that $\Psi_2^{-1/3}$ can be moved inside the $\{\}_B$ brackets in view of the background Bianchi identities (2.12).

Combining the above identities as $(\text{þ}-2\rho-\bar{\rho})(3.17) + (\text{ð}-2\tau-\bar{\tau}')(3.18)$, we get

$$\left[(\text{þ}-2\rho-\bar{\rho})(\text{þ}'-\rho') - (\text{ð}-2\tau-\bar{\tau}')(\text{ð}'-\tau')\right](\Psi_2^{-1/3}\Psi_{1B}) =$$

$$\frac{3}{2}\left\{\left[(\text{þ}-2\rho-\bar{\rho})(\text{ð}-2\tau) - (\text{ð}-2\tau-\bar{\tau}')(\text{þ}-2\rho)\right]\Psi_2^{2/3}\right\}_B.$$

which gives

$$\left[\boxdot_2 - 4\Psi_2\right]\left(\Psi_2^{-1/3}\Psi_{1B}\right) = 3\left\{\left[(\text{þ}-2\rho-\bar{\rho})(\text{ð}-2\tau) - (\text{ð}-2\tau-\bar{\tau}')(\text{þ}-2\rho)\right]\Psi_2^{2/3}\right\}_B,$$

(3.19)

using (3.5). Expanding the right hand side of equation (3.19), leaving off a factor of 3, we have

$$\left[(\text{þ}-2\rho-\bar{\rho})(\text{ð}-2\tau) - (\text{ð}-2\tau-\bar{\tau}')(\text{þ}-2\rho)\right]_B\Psi_2^{2/3}+$$

$$+\left[(\text{þ}-2\rho-\bar{\rho})(\text{ð}-2\tau) - (\text{ð}-2\tau-\bar{\tau}')(\text{þ}-2\rho)\right]\left[\Psi_2^{2/3}\right]_B.$$

The second term vanishes due to the commutation relation (2.13) with $a=2$, but for the first term, we must use identites valid off the background. Since Ψ_2 is a $\{0,0\}$ quantity, we find

$$\left[(\text{þ}-2\rho-\bar{\rho})(\text{ð}-2\tau) - (\text{ð}-2\tau-\bar{\tau}')(\text{þ}-2\rho)\right]_B\Psi_{2A}^{2/3} =$$

$$= \left[-\bar{\tau}'\,\text{þ}-\kappa\text{þ}' + \bar{\rho}\,\text{ð}+\sigma\text{ð}' - 2\tau\,\text{þ}-2(\text{þ}\,\tau)+\right.$$

$$\left. + 2\rho\,\text{ð}+2(\text{ð}\,\rho) - (2\rho+\bar{\rho})(\text{ð}-2\tau) + (2\tau+\bar{\tau}')(\text{þ}-2\rho)\right]_B\Psi_{2A}^{2/3} \quad (3.20)$$

$$= \left[-2\kappa\rho' + 2\sigma\tau' - 2(\text{þ}\,\tau) + 2(\text{ð}\,\rho) + 2\tau\bar{\rho} - 2\bar{\tau}'\rho\right]_B\Psi_{2A}^{2/3}$$

$$= -2\Psi_{2A}^{2/3}\left[(\text{þ}'+2\rho' - \bar{\rho}')\kappa_B - (\text{ð}'+2\tau' - \bar{\tau})\sigma_B + 2\Psi_{1B}\right],$$

where we used the commutation relation (2.11b) in the first step, background Bianchi identities in the second step and the Ricci identities (2.8a)$'$ and (2.8a) in the last step. Applying a prime and making use of (3.6) gives (3.11).

Finally we consider the spin weight 0 linearized Weyl scalar Ψ_{2B}. Recall that Ψ_2 is non-vanishing in a type D spacetime. For this reason, it is convenient in the calculations to leave some expressions as $\{\}_B$ brackets. As in the previous cases,

we start with linearized Bianchi identities. From $(2.9b)'$ and $(2.9b)^{*\prime}$, we get

$$\left\{(\textrm{þ}' - 3\rho')\Psi_2\right\}_B = (\eth - 2\tau)\Psi_{3B},$$

$$-\left\{(\eth' - 3\tau')\Psi_2\right\}_B = -(\textrm{þ} - 2\rho)\Psi_{3B}. \tag{3.21}$$

Multiplying both equations by $\Psi_2^{-2/3}$ and using the Leibniz rule gives rescaled equations

$$\left\{3(\textrm{þ}' - \rho')\Psi_2^{1/3}\right\}_B = \Psi_2^{-2/3}(\eth - 2\tau)\Psi_{3B}, \tag{3.22}$$

$$-\left\{3(\eth' - \tau')\Psi_2^{1/3}\right\}_B = -\Psi_2^{-2/3}(\textrm{þ} - 2\rho)\Psi_{3B}. \tag{3.23}$$

Here we used the fact that $\Psi_2^{-2/3}$ can be moved inside the $\{\}_B$ brackets because of the background Bianchi identities (2.12).

To find a wave equation for Ψ_{2B}, we consider the combination $(\textrm{þ} - \bar{\rho})(3.22) + (\eth - \bar{\tau}')(3.23)$, which gives

$$(\textrm{þ} - \bar{\rho})\left\{3(\textrm{þ}' - \rho')\Psi_2^{1/3}\right\}_B - (\eth - \bar{\tau}')\left\{3(\eth' - \tau')\Psi_2^{1/3}\right\}_B =$$
$$(\textrm{þ} - \bar{\rho})\left(\Psi_2^{-2/3}(\eth - 2\tau)\Psi_{3B}\right) - (\eth - \bar{\tau}')\left(\Psi_2^{-2/3}(\textrm{þ} - 2\rho)\Psi_{3B}\right). \tag{3.24}$$

Using the identities $(\textrm{þ} - \bar{\rho})(\Psi_2^{-2/3}\phi) = \Psi_2^{-2/3}(\textrm{þ} - 2\rho - \bar{\rho})\phi$ and $(\eth - \bar{\tau}')(\Psi_2^{-2/3}\phi) = \Psi_2^{-2/3}(\eth - 2\tau - \bar{\tau}')\phi$, which follow from (2.12), the RHS vanishes due to (2.13) with $a = 2$. The operators on the LHS can be put into the $\{\}_B$ brackets because of the background Bianchi identities (2.12). This gives the identity

$$\left\{\left[\boxdot_0 + 2\Psi_2\right]\Psi_2^{1/3}\right\}_B = 0. \tag{3.25}$$

Expanding the $\{\}_B$ bracket in equation (3.25) gives

$$0 = 3\left\{\left[\boxdot_0 + 2\Psi_2\right]\Psi_2^{1/3}\right\}_B \tag{3.26a}$$

$$= \left[\Box + 2\Psi_2\right]\left(\Psi_2^{-2/3}\Psi_{2B}\right) + 3\left[\Box_B + 2\Psi_{2B}\right]\Psi_2^{1/3} \tag{3.26b}$$

$$= \left[\Box + 8\Psi_2\right]\left(\Psi_2^{-2/3}\Psi_{2B}\right) + 3\Box_B\Psi_2^{1/3}. \tag{3.26c}$$

and hence (3.10). This completes the proof.   □

For future reference, we state the following equations which were used in the proof of theorem 3.3

Corollary 3.4.

$$\left[\boxdot_2 - 4\Psi_2\right]\left(\Psi_2^{-1/3}\Psi_{1B}\right) = 3\left\{\left[(\textrm{þ} - 2\rho - \bar{\rho})(\eth - 2\tau) - (\eth - 2\tau - \bar{\tau}')(\textrm{þ} - 2\rho)\right]\Psi_2^{2/3}\right\}_B, \tag{3.27a}$$

$$\left\{\left[\boxdot_0 + 2\Psi_2\right]\Psi_2^{1/3}\right\}_B = 0, \tag{3.27b}$$

$$\left[\boxdot_2 - 4\Psi_2\right]\left(\Psi_2^{-1}\Psi_{3B}\right) = 3\Psi_2^{-2/3}\left\{\left[(\textrm{þ} - 2\rho - \bar{\rho})(\eth - 2\tau) - (\eth - 2\tau - \bar{\tau}')(\textrm{þ} - 2\rho)\right]\Psi_2^{2/3}\right\}_B', \tag{3.27c}$$

The rescaled Bianchi identity (3.18) has the same form as the perturbed Maxwell equation (2.10) on a charged type D background ($\phi_{1A} \neq 0$)

$$\left\{ (\text{\th} - 2\rho)\phi_1 \right\}_B = (\eth' - \tau')\phi_{0B}. \tag{3.28}$$

Therefore the decoupled electromagnetic perturbation equations follow immediately, and we have the following result.

Corollary 3.5. *In a charged type D background the Maxwell components $\phi_{iB}, i = 0, 1, 2$ fulfill the equations*

$$\left[\boxdot_2 - 4\Psi_2\right]\phi_{0B} = 2\left\{ \left[(\text{\th} - 2\rho - \bar{\rho})(\eth - 2\tau) - (\eth - 2\tau - \bar{\tau}')(\text{\th} - 2\rho)\right]\phi_1 \right\}_B \tag{3.29}$$

$$\left\{ \left[(\text{\th} - \rho - \bar{\rho})(\text{\th}' - 2\rho') - (\eth - \tau - \bar{\tau}')(\eth' - 2\tau')\right]\phi_1 \right\}_B = 0 \tag{3.30}$$

$$\left[\boxdot_2' - 4\Psi_2\right]\phi_{2B} = 2\left\{ \left[(\text{\th} - 2\rho - \bar{\rho})(\eth - 2\tau) - (\eth - 2\tau - \bar{\tau}')(\text{\th} - 2\rho)\right]'\phi_1 \right\}_B \tag{3.31}$$

For a charged type D spacetime, the background Bianchi identities (2.9) include a term involving $\phi_1\bar{\phi}_1$ and hence the simple rescaling used above for vacuum type D backgrounds does not apply. Instead one can use the background Maxwell equations (2.10), namely

$$\text{\th}\,\phi_1 = 2\rho\phi_1, \quad \eth\,\phi_1 = 2\tau\phi_1,$$

as in [18]. But as the Ψ_2 rescaling is singular for the limit of flat background, the ϕ_1 rescaling does not work for the limit of uncharged background.

In the special case of a test Maxwell field on an uncharged background, rescaling by Ψ_2 becomes possible and the equation for ϕ_{1B} reduces to the Fackerell-Ipser equation [19]

$$\left[\Box + 2\Psi_2\right](\Psi_2^{-1/3}\phi_{1B}) = 0,$$

while the equations for ϕ_{0B} and ϕ_{2B} become the spin $s = \pm 1$ TME$_s$.

4. Gauge source functions

In this section we consider the equations for the gauge dependent quantities $\Psi_{1B}, \Psi_{2B}, \Psi_{3B}$ in more detail.

4.1. **Gauge source functions for the Einstein equations.** In [23], Friedrich derived a frame based, symmetric hyperbolic system for the Einstein-Yang-Mills system. We specialize to the vacuum case and set the conformal factor $\Omega = 1$. Then, the result of [23] gives a symmetric hyperbolic system for a set of unknowns consisting of a null tetrad (or spin frame), the spin coefficients and the Weyl spinor.

Starting from a system of equations involving the tetrad and the curvature components as variables, he identified the gauge source functions for this system. These are, letting (x^μ) be coordinates on M,

$$F^\mu = \Box x^\mu,$$

and

$$F_{\mathfrak{ab}} = \nabla^a(\nabla_a e_{\mathfrak{a}}^b)e_{\mathfrak{b}}^c g_{bc},$$

where $(e_{\mathfrak{a}})$ is a null tetrad, cf. [23, equations (2.6), (2.13)], see also [24].

For the frame based hyperbolic system considered by Friedrich one may freely specify the gauge source functions as functions of the spacetime coordinate, the tetrad, the connection coefficinets, and the Weyl tensor components, i.e.

$$F^\mu = F^\mu(x^\alpha, e_{\mathfrak{a}}^\alpha, \Gamma_{\mathfrak{bc}}^{\mathfrak{a}}, W_{\mathfrak{abcd}}), \tag{4.1}$$

$$F_{\mathfrak{ab}} = F_{\mathfrak{ab}}(x^\alpha, e_{\mathfrak{a}}^\alpha, \Gamma_{\mathfrak{bc}}^{\mathfrak{a}}, W_{\mathfrak{abcd}}), \tag{4.2}$$

without changing the principal part of the resulting symmetric hyperbolic system, see the discussion in [24], in particular [24, p. 1462]. As we are in a geometric situation where it is natural to adapt to a specific background geometry and to use a GHP weighted tetrad, it is convenient to consider the following modified gauge source functions. They differ from the expressions given by Friedrich by lower order terms, which do not change the principal part of the resulting reduced system.

To define the coordinate gauge source function, fix a background metric $\hat{g}_{ab}$ on M, with Levi-Civita derivative $\widehat{\nabla}_a$ and let V^a be the tension field for the identity map $(M, g_{ab}) \to (M, \hat{g}_{ab})$ defined by

$$V^a \xi_a = g^{cd}(\widehat{\nabla}_c - \nabla_c)\xi_d,$$

holds for any 1-form ξ_a, see [2] for details. Then a gauge source function for the coordinate degrees of freedom can be given by the equation

$$F^\mu = V^\mu. \tag{4.3}$$

Further, let Θ_a be the weighted GHP covariant derivative and let $(e_{\mathfrak{a}})$ be a weighted GHP tetrad. A gauge source function for the tetrad degrees of freedom can be given by the equation

$$F_{\mathfrak{a}\mathfrak{b}} = \Theta^a(\Theta_a e_{\mathfrak{a}}^b)e_{\mathfrak{b}}^c g_{bc}. \tag{4.4}$$

These are the expressions which we shall consider below.

4.1.1. *Gauge source functions for the Linearized Einstein equations.* Consider linearized perturbations around vacuum spacetime (M, g_{ab}). The work of Friedrich on hyperbolic reductions carries over immediately to the linearized vacuum field equations. Thus we may consider linearized frame based systems with unknowns consisting of the linearized tetrad, the linearized spin coefficients and the linearized Weyl scalars. The reduced system is extracted by specifying linearized coordinate and frame gauge source functions F_B^μ, $F_{B\mathfrak{a}\mathfrak{b}}$ which may be specified freely as functions of the unknowns which are linear as functions of $e_{B\mathfrak{a}}^\alpha, \Gamma_{B\mathfrak{b}\mathfrak{c}}^{\mathfrak{a}}, W_{B\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}}$. Thus, with this restriction, we may consider gauge source functions

$$F_B^\mu = F_B^\mu(x^\alpha, e_{B\mathfrak{a}}^\alpha, \Gamma_{B\mathfrak{b}\mathfrak{c}}^{\mathfrak{a}}, W_{B\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}}),$$
$$F_{B\mathfrak{a}\mathfrak{b}} = F_{B\mathfrak{a}\mathfrak{b}}(x^\alpha, e_{B\mathfrak{a}}^\alpha, \Gamma_{B\mathfrak{b}\mathfrak{c}}^{\mathfrak{a}}, W_{B\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}}).$$

4.2. **Linearized Weyl scalars and gauge.** In the following discussion it is in some steps convenient to use a δ to denote first order linearized fields. In particular, $\delta g_{ab} = h_{ab}$ and denote the resulting perturbations in geometric fields defined in terms of g_{ab} by e.g. δR_{ab}. We have

$$\delta R_{ab} = -\frac{1}{2}\Box h_{ab} - R_a{}^c{}_b{}^d h_{cd} + \nabla_{(a}v_{b)},$$

where, letting $h = g^{ab}h_{ab}$, and working in a coordinate system (x^μ),

$$v^\alpha = \nabla_\beta h^{\beta\alpha} - \frac{1}{2}\nabla^\alpha h = g^{\beta\nu}\delta\Gamma_{\beta\nu}^\alpha.$$

The vector field v^α defined by this expression is precisely the linearization of the tension field V^α around g_{ab} (playing the role of the background metric $\hat{g}_{ab}$ above). Thus v^α is the appropriate coordinate gauge source function for linearized perturbations $h_{\alpha\beta}$ of $g_{\alpha\beta}$ and the gauge condition which corresponds to (4.3) is given by

$$v^\alpha = F_B^\alpha.$$

The standard harmonic gauge, also known as deDonder gauge, with respect to the background metric is given by the condition $F_B^\alpha = 0$. In this gauge, the linearized

Einstein equations in terms of the linearized metric h_{ab} take the form of a wave equation

$$\Box h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} = 0,$$

where $\Box = \nabla^c \nabla_c$ is the covariant d'Alembertian.

Let (M, g) be a vacuum type D spacetime. Working in a principal null tetrad, let $\psi = \Psi_2^{1/3}$. Recall that Ψ_2 is of type $\{0, 0\}$ and is thus a well-defined function on spacetime. In notation used in this section, the condition that the right hand side of (3.10) vanishes, i.e.,

$$\Box_B \Psi_2^{1/3} = 0,$$

takes the form

$$(\delta\Box)\psi = 0.$$

We have

$$(\delta\Box)\psi = -h^{ab}\nabla_a\nabla_b\psi - v^a\nabla_a\psi. \tag{4.5}$$

We can view this equation as specifying part of the coordinate gauge degrees of freedom. In the Schwarzschild case, working in a principal tetrad, ψ is real, and hence (4.5) specifies one component of v^a. In the general case, ψ is complex, while v^a is real. Taking the real and imaginary parts of (4.5) gives two real equations for v^a.

In order to analyze this equation in the Kerr case, it is convenient to calculate in a coordinate system and tetrad which is non-singular on the horizon. A tetrad in the ingoing Kerr coordinate system (also known as ingoing Eddington-Finkelstein coordinates) was described by Teukolsky [49, §5]. In this tetrad, the components $\nabla_{\mathfrak{a}}\psi$ are non-vanishing on the horizon.

As mentioned above, it is compatible with the well-posedness of the reduced field equations to allow the gauge source functions to depend on the Weyl scalars. Thus we may also consider gauge conditions of the form

$$-3\Box_B \Psi_2^{1/3} - 6\Psi_2^{1/3}\Psi_{2B} = 0, \tag{4.6}$$

which leads to the wave equation

$$(\Box + 2\Psi_2)(\Psi_2^{-2/3}\Psi_{2B}) = 0, \tag{4.7}$$

for Ψ_{2B}. Thus, in the gauge given by (4.6), $\Psi_2^{-2/3}\Psi_{2B}$ satisfies the Fackerell-Ipser equation. This substantiates the discussion in the work of Crossman and Fackerell [14, 18]. In the Kerr case, calculation shows that the gauge source function given by (4.6) will have terms depending on $1/a$, and hence this gauge condition behaves in a singular manner in the Schwarzschild limit. Equation (4.7) is not known to be separable or admit a symmetry operator, see however [18, p. 617]. This discussion shows that in the rotating case, also a generalized harmonic gauge condition leading to a homogenous wave equation

$$\Box(\Psi_2^{-2/3}\Psi_{2B}) = 0,$$

(which admits symmetry operators) is compatible with a well-posed reduced system.

Next we consider the phantom gauge condition. A calculation shows that

$$\Theta^a m^b \Theta_a l_b = 2[\flat'\kappa - \eth'\sigma - \bar{\rho}'\kappa + \bar{\tau}\sigma + \Psi_1].$$

Restricting to a type D background this expression vanishes, and the first order linearization gives

$$F_{Blm} := \frac{1}{2}\left\{\Theta^a m^b \Theta_a l_b\right\}_B = [(\flat' - \bar{\rho}')\kappa_B - (\eth' - \bar{\tau})\sigma_B + \Psi_{1B}]. \tag{4.8}$$

Now we can write the equation for Ψ_{1B} in the form

$$[\boxdot_2 - 4\Psi_2](\Psi_2^{-1/3}\Psi_{1B}) = -6\Psi_{2A}^{2/3}[F_{Blm} + 2\rho'\kappa_B + 2\tau'\sigma_B + \Psi_{1B}].$$

Thus, Chandrasekhar's phantom gauge condition can be written in terms of the gauge source function as

$$F_{Blm} = -2\rho'\kappa_B - 2\tau'\sigma_B - \Psi_{1B}.$$

In view of the discussion above, this form of the gauge source function is compatible with a hyperbolic system for the linearized Einstein equations. The equation for Ψ_{3B} can be handled along the same lines.

5. Gauge invariant equations on Schwarzschild background

Price [42] has shown that for linearized gravity on a Schwarzschild background, $r^3\mathfrak{Im}\Psi_{2B}$ describes odd parity perturbations. He used the special coordinates of Newman and Penrose [36, p.572] for the perturbed spacetime, which can be understood as a coordinate gauge. Price expressed the perturbed spin coefficients, which occur in the Ψ_{2B} equation, in terms of perturbed metric coefficients. These coefficients are real in the odd parity case and therefore cancel. Price then used the definition of $\mathfrak{Im}\Psi_{2B}$ in terms of perturbed Riemann tensor components, to relate it to the perturbed metric coefficients. He showed that these coincide up to a time derivative, in Regge-Wheeler (RW) gauge, with the RW variable Q for odd parity perturbations.

Starting from (3.26c),

$$\left[\Box + 8\Psi_2\right]\left(\Psi_2^{-2/3}\Psi_{2B}\right) = -3\Box_B\Psi_2^{1/3}, \tag{5.1}$$

we rederive the known result, that the imaginary part satisfies the gauge invariant RW equation [35], which reduces to the result of Price in RW gauge. We also show that the real part gives the gauge invariant Zerilli equation. Price was not able to derive this equation, possibly due to his choice of special coordinates.

The conventions of Martel and Poisson [34] will be used in this section. Indices $a, b, \ldots$ for coordinates t, r and $A, B, \ldots$ for coordinates θ, φ. This notation differs from the convention used by Regge-Wheeler and Zerilli in that $K_{\mathrm{MP}} = K_{\mathrm{RW}} - \frac{1}{2}l(l+1)G$. Metric perturbations are denoted $g_{\mu\nu} = g_{A\mu\nu} + p_{\mu\nu}$, since $h_{\mu\nu}$ is used for some spherical harmonic decomposed components. It should be noted that we still use the signature $(+---)$ while Martel and Poisson use the signature $(-+++)$. Using Schwarzschild coordinates $g_{ab} = \mathrm{diag}(f, -f^{-1}, -r^2, -r^2\sin^2\theta)$, where $f = 1 - \frac{2M}{r}$ and $\Psi_2 = -M/r^3$, the Kinnersley frame reads

$$l^a = \left(f^{-1}, 1, 0, 0\right), \quad n^a = \frac{1}{2}\left(1, -f, 0, 0\right), \quad m^a = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right).$$

With $\psi = \Psi_2^{1/3}$, the RHS of (5.1) reduces to

$$
\begin{aligned}
\Box_B\psi \overset{(4.5)}{=} & -p^{\mu\nu}\nabla_{\mu\nu}^2\psi - \left(\nabla^\mu p^\sigma_{\;\mu} - \frac{1}{2}\nabla^\sigma p_\mu^{\;\mu}\right)\nabla_\sigma\psi \\
= & \left(-p^{\mu r}\partial_\mu - (\partial_\mu p^{r\mu}) - p^{r\rho}\Gamma^\mu_{\mu\rho} + g^{rr}(\partial_r p_\mu^{\;\mu})\right)\partial_r\psi.
\end{aligned}
\tag{5.2}
$$

Further simplifications will occur in odd and even part, which are investigated in the next sections.

To relate Ψ_{2B} to Regge-Wheeler and Zerilli variables we do a calculation analogous to that of Price [42, Appendix D]. Starting from

$$
\begin{aligned}
-2\Psi_{2B} = & R_{A\alpha\beta\gamma\delta}(l^\alpha n^\beta l^\gamma n^\delta)_B - R_{A\alpha\beta\gamma\delta}(l^\alpha n^\beta m^\gamma \bar{m}^\delta)_B \\
& + R_{B\alpha\beta\gamma\delta}(l^\alpha n^\beta l^\gamma n^\delta)_A - R_{B\alpha\beta\gamma\delta}(l^\alpha n^\beta m^\gamma \bar{m}^\delta)_A,
\end{aligned}
\tag{5.3}
$$

we get

$$-2\Psi_{2B} = R_{Btrtr} + \frac{i}{r^2\sin\theta}R_{Brt\theta\phi} + \frac{4M}{r^3}p_{\alpha\beta}l^\alpha n^\beta. \tag{5.4}$$

The second term is purely imaginary and known from the calculations of Price. The perturbed Riemann tensor is related to metric perturbations $p_{\mu\nu}$ via

$$R_{B\alpha\beta\gamma\delta} = \tfrac{1}{2}(p_{\beta\gamma;\alpha\delta} + p_{\alpha\delta;\beta\gamma} - p_{\alpha\gamma;\beta\delta} - p_{\beta\delta;\alpha\gamma} + R_{A\alpha\sigma\gamma\delta}p_\beta^\sigma + R_{A\sigma\beta\gamma\delta}p_\alpha^\sigma). \quad (5.5)$$

With these relations one can check explicitly that $\mathfrak{Im}\Psi_{2B}$ corresponds to odd parity and $\mathfrak{Re}\Psi_{2B}$ corresponds to even parity perturbations.

5.1. **Imaginary part and Regge Wheeler equation.** The odd parity metric perturbation can be expressed as [34, eq. 5.1 - 3]

$$p_{ab} = 0, \quad p_{aB} = \sum_{lm} h_a^{lm} X_B^{lm}, \quad p_{AB} = \sum_{lm} h_2^{lm} X_{AB}^{lm},$$

where X_B^{lm} and X_{AB}^{lm} are vector and tensor spherical harmonics. It follows that $\operatorname{tr} p_{\mu\nu} = 0$ and after a short calculation (5.2) reduces to

$$\Box_B \psi = 0.$$

$\mathfrak{Im}\Psi_{2B}$ corresponds to odd parity perturbations and $\mathfrak{Im}(5.1)$ reduces in this case to

$$\left[\Box + 8\Psi_2\right]\left(\Psi_2^{-2/3}\mathfrak{Im}\Psi_{2B}\right) = 0.$$

Introducing tortoise coordinates r_* by $\partial_{r*} = f\partial_r$ cancels $2\Psi_2$ from the potential, a rescaling by r cancels first order ∂_r terms and we are left with a gauge invariant Regge Wheeler equation

$$\left[\partial_t^2 - \partial_{r_*}^2 + f\frac{l(l+1)}{r^2} - f\frac{6M}{r^3}\right](r^3\mathfrak{Im}\Psi_{2B}) = 0.$$

To relate this to the Regge Wheeler variable Q, we look at (5.4). For odd parity perturbations it gives $(R_B^{odd})_{trtr} = 0 = p_{\alpha\beta}^{\text{odd}}l^\alpha n^\beta$ and with (5.5),

$$\frac{R_{Brt\theta\phi}^{odd}}{r^2\sin\theta} = \frac{l(l+1)}{2}\left[\left(\frac{h_t}{r^2}\right)_{,r} - \left(\frac{h_r}{r^2}\right)_{,t}\right],$$

(for convenience we suppress spherical harmonics and the related indices). It follows that just the imaginary part contributes to the perturbations. We now have

$$-r^3\mathfrak{Im}\Psi_{2B,t} = \frac{r^3}{4}l(l+1)\left[\left(\frac{h_t}{r^2}\right)_{,r} - \left(\frac{h_r}{r^2}\right)_{,t}\right]_{,t}$$

$$= \frac{f(l+2)!}{4r(l-2)!}\left[h_r + \frac{1}{2}h_{2,r} - \frac{1}{r}h_2\right] = \frac{(l+2)!}{4(l-2)!}Q^{\text{odd}},$$

which is the gauge invariant variable of Moncrief [35] and in RW gauge reduces to the result of Price [42].

5.2. **Real part and Zerilli equation.** The even parity metric perturbations can be written [34, eq. 4.1 - 3]

$$p_{ab} = \sum_{lm} h_{ab}^{lm} Y^{lm}, \quad p_{aB} = \sum_{lm} j_a^{lm} Y_B^{lm}, \quad p_{AB} = r^2 \sum_{lm} K^{lm}\Omega_{AB}Y^{lm} + G^{lm}Y_{AB}^{lm},$$

$$(5.6)$$

where Y^{lm}, Y_B^{lm} and Y_{AB}^{lm} denotes the even parity scalar, vector and tensor spherical harmonics. $\mathfrak{Re}\Psi_{2B}$ is not gauge invariant, but transforms as

$$\Psi_{2B} \to \Psi_{2B} - \frac{3M}{r^4}\xi^r.$$

For some coordinate gauge transformation $x^\mu \to x^\mu + \xi^\mu$. Therefore we rewrite the wave equation for the gauge invariant quantity, using Appendix A

$$\widetilde{\Psi}_{2B} := r^3 \mathfrak{Re}\Psi_{2B} + \frac{3M}{2}\left(K + \frac{\lambda}{2}G\right), \tag{5.7}$$

where $\lambda = l(l+1)$. The Ψ_{2B} equation (5.1) for even parity perturbations reads

$$\left[\partial_t^2 - \partial_{r_*}^2 + f\frac{\lambda}{r^2} - f\frac{6M}{r^3}\right]\widetilde{\Psi}_{2B} =$$
$$3Mfr\Box_B r^{-1} + \frac{3M}{2}\left[\partial_t^2 - \partial_{r_*}^2 + f\frac{\lambda}{r^2} - f\frac{6M}{r^3}\right]\left(K + \frac{\lambda}{2}G\right). \tag{5.8}$$

The perturbed wave operator term (5.2) does not vanish, but gives

$$\Box_B\frac{1}{r} = -\frac{1}{r^2}\left[\frac{\lambda f}{r^2}j_r - \left(\frac{3Mf}{r^2} + \frac{f^2}{2}\partial_r\right)h_{rr} + \partial_t h_{tr} + \left(\frac{M}{fr^2} - \frac{1}{2}\partial_r\right)h_{tt} + f\partial_r K\right]. \tag{5.9}$$

Expanding (5.4) for even parity perturbations gives after some calculations $R_{Brt\theta\phi} = 0$ and

$$\mathfrak{Re}\Psi_{2B} = M(f^{-1}h_{tt} - fh_{rr})$$
$$+ \frac{r^3}{4}\left[\partial_r^2 h_{tt} + \partial_t^2 h_{rr} - \frac{m}{r^2}\partial_r(f^{-1}h_{tt} - fh_{rr}) - \frac{2m}{r^2 f}\partial_t h_{tr} - 2\partial_{tr}^2 h_{tr}\right]. \tag{5.10}$$

The whole equation is now given in terms of metric perturbations. To compare the result to others, we express all metric components in terms of gauge invariants (denote with a tilde according to the conventions of Martel and Poisson). Using the equations of Appendix A, $\widetilde{\Psi}_{2B}$ takes the form

$$\widetilde{\Psi}_{2B} = \frac{\Lambda r}{4}\left[\widetilde{K} + \frac{2f}{\Lambda}\left(f\widetilde{h}_{rr} - r\partial_r\widetilde{K}\right)\right], \tag{5.11}$$

where $\lambda = l(l+1)$, $\mu = (l-1)(l+2) = \lambda - 2$, $\Lambda = \mu + 6M/r$. The RHS of (5.8) can also be expressed in terms of gauge invariant quantities

$$\text{RHS} = -f\frac{6M}{r^3}\widetilde{\Psi}_{2B} + \frac{3Mf\lambda}{2r^2}\widetilde{K}. \tag{5.12}$$

The first term cancels the RW potential on the LHS! Now we rescale (5.8) by Λ^{-1} (which depends on r),

$$\left[\partial_t^2 - \partial_{r_*}^2 + f\frac{\lambda}{r^2} + \Lambda(\partial_{r_*}^2\Lambda^{-1})\right](\Lambda^{-1}\widetilde{\Psi}_{2B}) = \frac{3Mf\lambda}{2r^2}\Lambda^{-1}\widetilde{K} - 2(\partial_{r_*}\Lambda^{-1})(\partial_{r_*}\widetilde{\Psi}_{2B}). \tag{5.13}$$

A straight forward but tedious calculation shows that the RHS can be written as

$$\frac{3Mf\lambda}{2r^2}\Lambda^{-1}\widetilde{K} - 2(\partial_{r_*}\Lambda^{-1})(\partial_{r_*}\widetilde{\Psi}_{2B}) = f\frac{6M}{r^3}\frac{\lambda}{\Lambda}\Lambda^{-1}\widetilde{\Psi}_{2B}, \tag{5.14}$$

the new potential term on the LHS simplifies to

$$\Lambda(\partial_{r_*}^2\Lambda^{-1}) = \frac{12Mf}{\Lambda^2 r^4}\left(\frac{6M^2}{r} + 3M\mu - \mu r\right), \tag{5.15}$$

and we finally end up with

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{12Mf}{\Lambda^2 r^4}\left(\frac{6M^2}{r} + 3\mu M + \frac{\mu^2 r}{2} + \frac{\mu^2 r^2}{6M}\left(\frac{\mu}{2} + 1\right)\right)\right](\Lambda^{-1}\widetilde{\Psi}_{2B}) = 0. \tag{5.16}$$

This is the gauge invariant Zerilli equation. The relation to Moncrief's gauge invariant variable is simply $Q^{\text{even}} = 4\Lambda^{-1}\widetilde{\Psi}_{2B}$.

Appendix A. Even partity perturbations in Schwarzschild coordinates

We used equations of the unpublished appendix of [34], which is available as arXiv:gr-qc/0502028. For convenience we repeat the required results of appendix C.

The even-parity metric perturbations (5.6) transform under even parity gauge $\xi_a = (\xi_t^{lm} Y^{lm}, \xi_r^{lm} Y^{lm}, \xi^{lm} Y_A^{lm})$ as

$$\delta h_{tt} = -2\frac{\partial}{\partial t}\xi_t + \frac{2Mf}{r^2}\xi_r, \qquad\qquad \delta h_{tr} = -\frac{\partial}{\partial r}\xi_t - \frac{\partial}{\partial t}\xi_r + \frac{2M}{r^2 f}\xi_t,$$

$$\delta h_{rr} = -2\frac{\partial}{\partial r}\xi_r - \frac{2M}{r^2 f}\xi_r, \qquad\qquad \delta j_t = -\frac{\partial}{\partial t}\xi - \xi_t,$$

$$\delta j_r = -\frac{\partial}{\partial r}\xi - \xi_r + \frac{2}{r}\xi, \qquad\qquad \delta K = -\frac{2f}{r}\xi_r + \frac{\lambda}{r^2}\xi,$$

$$\delta G = -\frac{2}{r^2}\xi.$$

Martel and Poisson extracted the following gauge invariant quantities

$$\tilde{h}_{tt} = h_{tt} - 2\frac{\partial}{\partial t}j_t + \frac{2Mf}{r^2}j_r + r^2\frac{\partial^2}{\partial t^2}G - Mf\frac{\partial}{\partial r}G,$$

$$\tilde{h}_{tr} = h_{tr} - \frac{\partial}{\partial r}j_t - \frac{\partial}{\partial t}j_r + \frac{2M}{r^2 f}j_t + r^2\frac{\partial^2}{\partial t\partial r}G + \frac{r-3M}{f}\frac{\partial}{\partial t}G,$$

$$\tilde{h}_{rr} = h_{rr} - 2\frac{\partial}{\partial r}j_r - \frac{2M}{r^2 f}j_r + r^2\frac{\partial^2}{\partial r^2}G + \frac{2r-3M}{f}\frac{\partial}{\partial r}G,$$

$$\tilde{K} = K - \frac{2f}{r}j_r + rf\frac{\partial}{\partial r}G + \frac{\lambda}{2}G,$$

and with these, the vacuum field equations are

$$0 = -\frac{\partial^2}{\partial r^2}\tilde{K} - \frac{3r-5M}{r^2 f}\frac{\partial}{\partial r}\tilde{K} + \frac{f}{r}\frac{\partial}{\partial r}\tilde{h}_{rr} + \frac{(\lambda+2)r+4M}{2r^3}\tilde{h}_{rr} + \frac{\mu}{2r^2 f}\tilde{K},$$

$$0 = \frac{\partial^2}{\partial t\partial r}\tilde{K} + \frac{r-3M}{r^2 f}\frac{\partial}{\partial t}\tilde{K} - \frac{f}{r}\frac{\partial}{\partial t}\tilde{h}_{rr} - \frac{\lambda}{2r^2}\tilde{h}_{tr},$$

$$0 = -\frac{\partial^2}{\partial t^2}\tilde{K} + \frac{(r-M)f}{r^2}\frac{\partial}{\partial r}\tilde{K} + \frac{2f}{r}\frac{\partial}{\partial t}\tilde{h}_{tr} - \frac{f}{r}\frac{\partial}{\partial r}\tilde{h}_{tt} + \frac{\lambda r+4M}{2r^3}\tilde{h}_{tt} - \frac{f^2}{r^2}\tilde{h}_{rr} - \frac{\mu f}{2r^2}\tilde{K},$$

$$0 = \frac{\partial}{\partial t}\tilde{h}_{rr} - \frac{\partial}{\partial r}\tilde{h}_{tr} + \frac{1}{f}\frac{\partial}{\partial t}\tilde{K} - \frac{2M}{r^2 f}\tilde{h}_{tr},$$

$$0 = -\frac{\partial}{\partial t}\tilde{h}_{tr} + \frac{\partial}{\partial r}\tilde{h}_{tt} - f\frac{\partial}{\partial r}\tilde{K} - \frac{r-M}{r^2 f}\tilde{h}_{tt} + \frac{(r-M)f}{r^2}\tilde{h}_{rr},$$

$$0 = -\frac{\partial^2}{\partial t^2}\tilde{h}_{rr} + 2\frac{\partial^2}{\partial t\partial r}\tilde{h}_{tr} - \frac{\partial^2}{\partial r^2}\tilde{h}_{tt} - \frac{1}{f}\frac{\partial^2}{\partial t^2}\tilde{K} + f\frac{\partial^2}{\partial r^2}\tilde{K}$$
$$+ \frac{2(r-M)}{r^2 f}\frac{\partial}{\partial t}\tilde{h}_{tr} - \frac{r-3M}{r^2 f}\frac{\partial}{\partial r}\tilde{h}_{tt} - \frac{(r-M)f}{r^2}\frac{\partial}{\partial r}\tilde{h}_{rr} + \frac{2(r-M)}{r^2}\frac{\partial}{\partial r}\tilde{K}$$
$$+ \frac{\lambda r^2 - 2(2+\lambda)Mr + 4M^2}{2r^4 f^2}\tilde{h}_{tt} - \frac{\lambda r^2 - 2\mu Mr - 4M^2}{2r^4}\tilde{h}_{rr},$$

$$0 = \frac{1}{f}\tilde{h}_{tt} - f\tilde{h}_{rr}.$$

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